Variational inference and dissipative adaptation

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We will construct a model of a self-organising system subject to dissipative fluxes wherein the system performs variational inference on the dynamics of its environment. At a basic level this will be a purely mathematical statement that the statistics of an environment can be 'read off' by a system situated inside it [1, 2], analogous to the celebrated framework of Jaynes at equilibrium [3]. For more sophisticated systems this can be thought of as a principle *by which* systems maintain non-equilibrium steady states [4], recapitulating such works as [5, 6, 7, 8, 9, 10, 11]. The author is grateful to K Dill and K Friston for many discussions around this topic.

A non-equilibrium system can be said to perform inference in the following sense. If a system exists at a state out of equilibrium it must reflect some meaningful properties of its environment relating to (for instance) the reservoirs of free energy and fluxes of heat in and out of the system. If a system stays close to an out of equilibrium state for long periods of time, it must be a better source of information about its environment; dually, one may argue that *self*-organising systems must store good models of their environment to better know what energy resources to take advantage of. Indeed, the main thesis of our framework is that any interacting systems capture the statistics of each others' probability densities *via* that interaction, and that estimating another system's statistics can be written as (variational) Bayesian inference.

Consider a large number of interacting particles (the system, a family of random variables $\{X_t\}_{t:\tau}$) coupled to a heat bath of constant temperature β^{-1}. Following [8, 9, 12] we will postulate the existence of a time-dependent field $\lambda(t)$ performing work on the system. Given a choice of $\lambda(t)$, the Itō process

$$\mathrm{d}X_t = b(X_t, \lambda_t)\,\mathrm{d}t + \varepsilon\sigma(X_t)\,\mathrm{d}W_t$$

generates trajectories γ with log-probability

$$\log p(\gamma^\dagger) - \beta\Delta Q(\gamma)$$

where $\gamma^\dagger$ is the time-reversed path. Our framework rests on an application of a theorem of Freidlin–Wentzell

$$\log p(\gamma \in \delta) = -\int_0^\tau \sigma^{-1}|\dot\gamma_t - b(\gamma_t, \lambda_t)|^2\,\mathrm{d}t$$

to leading order in ε, and consequently, that the expected path minimises $\log p(\gamma)$. (Note that this result can be extended to stochastic partial differential equations with some subtleties, covering active matter situations.) Consequently

$$\begin{aligned}\log p(\gamma^\dagger) &- \beta\Delta Q(\gamma) \\ &= -\int_0^\tau \sigma^{-1}|\dot\gamma_t - b(\gamma_t, \lambda_t)|^2\,\mathrm{d}t + o(\varepsilon). \quad (1)\end{aligned}$$

Observe the system has as a parameter the choice of $\lambda(t)$. We can imagine this as a control parameter or simply an abstract representation of the driving field. Following the argument in [2], this implies that for a system coupled to an environment in a particular way, the expected trajectory minimises the divergence between a parametric distribution over external states and a true density over external states.

If we imagine a system which modulates λ such that it evolves in a highly irreversible way, (1) yields an explicit estimate for how much heat must be dissipated in order to minimise fluctuations away from some desired time evolution. Relating this to a variational free energy function of probabilities over the environment gives us the notion that a system must be a 'good' representation of its environment as long as it consumes sufficiently large energetic resources to dissipate that amount of heat, modulating $\lambda(t)$ to weight outcome-likelihood ratios favourably. In the following lecture an explicit model of a non-equilibrium system satisfying this variational free energy principle will be given.

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[1] D A R Sakthivadivel, in *Active Inference: Third International Workshop* (Springer Nature, Cham, 2023), pp 298–318.

[2] M J D Ramstead, D A R Sakthivadivel, K J Friston. Preprint arXiv:2406.11630 (2024).

[3] E T Jaynes, Phys Rev **106**, 4 (1957).

[4] K J Friston, L Da Costa, D A R Sakthivadivel, C Heins, G A Pavliotis, M Ramstead, T Parr, Phys Life Rev **47** (2023)

[5] G Nicolis and I Prigogine, *Self-Organization in Nonequilibrium Systems* (Wiley, New York, 1977).

[6] S Still, D A Sivak, A J Bell, G E Crooks, Phys Rev Lett **109**, 12 (2012)

[7] J L England, Nat Nanotechnol **10** (2015).

[8] N Perunov, R A Marsland, J L England, Phys Rev X **6** (2016)

[9] U Seifert, Annu Rev Condens Matter Phys **10** (2019)

[10] T Isomura, K Kotani, Y Jimbo, K J Friston, Nat Commun **14** (2023).

[11] J A Patcher, Y J Yang, K A Dill, Nat Rev Phys **6** (2024)

[12] G E Crooks, Phys Rev E **60** (1999)